Year 12 Mathematics Term 2 Assessment 2004

QUESTION 1

(a) The population, P, of a town at time t years after the start of 1970 is estimated by the formula $P = P_0 e^{kt}$. Records show that the population of the town at the start of 1970 was 2916 and it had grown to 4860 by the start of 1980.

(i) Find the exact values for P_0 and k. 3

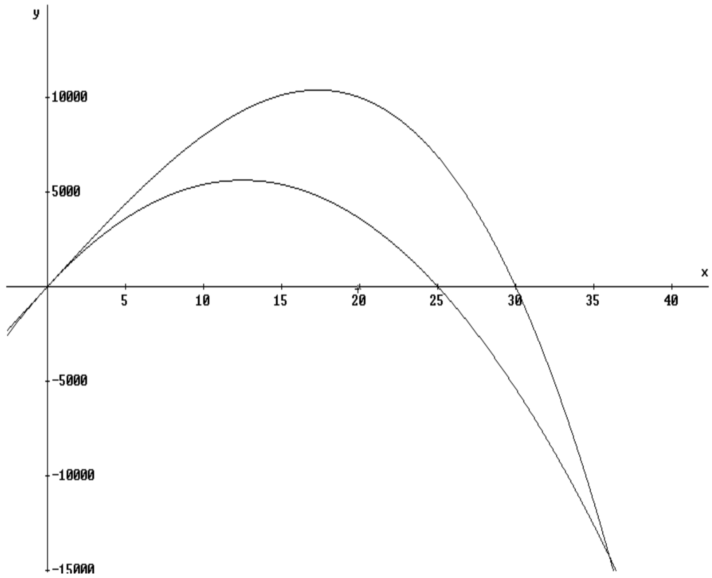
(ii) Find the population of the town at the start of 2010. (Give answer correct to nearest 100 people) 2

(iii) Prove that the rate of change of the population at time t years is given by $\dfrac{dP}{dt} = kP$ where k is a constant. 1

(iv) Find the rate of increase of the population
(α) when the population is 6000. (Give answer correct to 2 significant figures) 1
(β) at the beginning of the year 2000. (Give answer correct to 2 significant figures) 1

(b) The graph shows the positions $(x\ km)$ of two objects A and B at time t hours. The expressions for the positions are given by:

$$x_A = 900t - t^3$$
$$x_B = 900t - 36t^2$$



(i) Find the time interval in which the position of each object is to the right of the origin. 2

(ii) Find an expression (in terms of t) for the distance (D) between the two objects while they are both to the right of the origin 1

(iii) Find the greatest distance between the two objects while they are to the right of the origin 4

QUESTION 2 **(START A NEW PAGE)**

(a) Sally is given 3 Mars Bars. With each Mars Bar there is a 20% chance of winning a free Mars Bar.

(i) Draw a probability tree diagram for the above information. 1

What is the probability that Sally wins

(ii) no free Mars Bars? 1

(iii) at least 1 free Mars Bar? 1

(iv) exactly 1 free Mars Bar? 1

(b) (i) Sketch the parabola $y = 2x^2 + 5x - 12$ clearly showing all intercepts with the coordinate axes. 2

(ii) Hence or otherwise solve for p such that $2p^2 + 5p > 12$. 1

(c) An object initially at the origin moves with velocity ($v\ km/h$) given by $v = 24 + 10t - t^2$. Find

(i) the maximum speed of the object. 2

(ii) the acceleration when the object is at rest. 3

(iii) the total distance travelled by the object during the first 15 hours. 3

QUESTION 4 **(START A NEW PAGE)**

(a) The roots of the equation $3x^2 - 5x + 1 = 0$ are $x = \alpha$ and $x = \beta$. Find the value of

(i) $\alpha + \beta$. 1

(ii) $\alpha\beta$. 1

(iii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$. 2

(b) On a table are two jars. The red jar contains 5 cards numbered 0, 2, 4, 6 and 8 while the blue jar contains 5 cards numbered 1, 3, 5, 7 and 9. A card is drawn from each jar and the sum of the two cards is calculated.

(i) Draw a dot diagram illustrating the above information. 1

Find the probability that

(ii) the sum is more than 10. 1

(iii) the sum is a prime number. 1

(iv) the sum is more than 10 if it is known that the sum is prime. 2

(c) The acceleration ($\ddot{x}\ ms^{-2}$) of a particle at time t seconds is given by $\ddot{x} = \dfrac{-1}{(5t+1)^2}$. The particle is initially 3 metres to the left of the origin traveling with velocity 1 ms^{-1}.

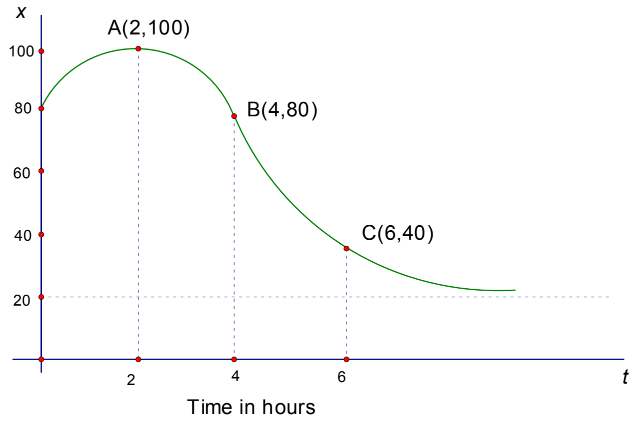
(i) Find an expression for the particle's velocity (v) at time t. 2

(ii) Find the limiting speed of the particle. 1

(iii) Determine the position of the particle at the end of the tenth second. (Give answer correct to the nearest metre) 3

QUESTION 3 **(START A NEW PAGE)**

(a) The displacement-time graph for an object is shown.
The graph has a turning point at $A(2,100)$ and an inflexion point at $B(4,80)$.



(i) Find when the object changes direction. 1

(ii) Find the position of the object when its speed is greatest. 1

(iii) Find the time interval for which the acceleration is negative. 2

(iv) Briefly describe the motion of the object between the points A and B. 2

(b) A scientist studying a penguin colony estimates that the number $N(t)$ of penguins in the colony at the end of t years is given by the formula $N(t) = \dfrac{A}{5+3e^{-0.1t}}$ where A is a constant.

(i) When the scientist starts her study the penguin population is estimated at 12 000. Find the value of A. 1

(ii) Find the estimated population of the colony at the end of 8 years. (Give your answer correct to the nearest 100 penguins) 1

(iii) Find the time required for the colony to grow to 18 000 penguins. (Give your answer correct to 1 decimal place) 2

(iv) Approximately how many penguins would you expect to find in the colony after a long time? 1

(c) (i) If the line $y = mx + b$ is tangent to the hyperbola $y = \dfrac{a}{x}$ prove that $b^2 + 4am = 0$. 2

(ii) Hence find the values of b for which the line $y = b - 3x$ will always intersect with the hyperbola $y = \dfrac{12}{x}$. 2

Term 2 2u 2004

Growth/Decay

Question 1

The population (P) of a town at time t years after the start of 1970 is estimated by the formula $P = P_0 e^{kt}$.
Records show that the population of the town at the start of 1970 was 2916 and it had grown to 4860 by the start of 1980.

(i) Find the exact values for P_0 and k.

(ii) Find the population of the town at the start of 2010.

(iii) Prove that the rate of change of the population at time t years is given by $\frac{dP}{dt} = kP$ where k is a constant.

(iv) Find the rate of increase of the population

(α) at the beginning of the year 2000.

(β) when the population is 6000.

Question 2

A scientist studying a penguin colony estimates that the number $N(t)$ of penguins in the colony at the end of t years is given by the formula $N(t) = \frac{A}{5 + 3e^{-0.1t}}$ where A is a constant.

(i) When the scientist starts her study the penguin population is estimated at 12 000. Find the value of A.

(ii) Find the estimated population of the colony at the end of 8 years. (Give your answer correct to the nearest 100 penguins)

(iii) Find the time required for the colony to grow to 18 000 penguins.

(iv) Approximately how many penguins would you expect to find in the colony after a long time?

Probability

Question 1

George has forgotten his 5 digit security number. He remembers that it is an odd number, no digits are repeated and it has alternating odd and even numbers.

(i) How many security numbers satisfy the above conditions?

(ii) Find the probability that his security number is greater than 85 000.

Question 2

Mary is given 3 Mars Bars. With each Mars Bar there is a 20% chance of winning a free Mars Bar.

(i) Draw a probability tree diagram for the above information.

What is the probability that Mary wins

(ii) no free Mars Bars?

(iii) at least 1 free Mars Bar?

(iv) exactly 1 free Mars Bar?

Question 3

On a table are two jars. The red jar contains 5 cards numbered 0, 2, 4, 6 and 8 while the blue jar contains 5 cards numbered 1, 3, 5, 7 and 9. A card is drawn from each jar and the sum of the two cards is calculated.

(i) Draw a dot/lattice diagram illustrating the above information.

Find the probability that

(ii) the sum is more than 10?

(iii) the sum is a prime number?

(iv) the sum is more than 10 if it is known that the sum is prime?

Motion

Question 1

An object initially at the origin moves with velocity ($v\ km/h$) given by $v = 24 + 10t - t^2$. Find

(i) the initial velocity of the object.
(ii) the maximum speed of the object.
(iii) the acceleration when the object is at rest.
(iv) the distance traveled by the object during the first 15 hours.

Question 2

The position (x m) of a particle at time t seconds is given by $x = 100t + 750e^{-0.05t}$.

(i) Find expressions for the velocity (v) and acceleration (a) at time t.

(ii) Find the initial position and velocity of the particle.

(iii) Find the time taken for the particle to reach a speed of 180 m/s.

(iv) Prove that $a = 0.05(100 - v)$.

Question 3

The position of an object at time t is given by $x = 3 + 4\sin^2 t$.

(i) Find expressions for its velocity and acceleration at any time t.

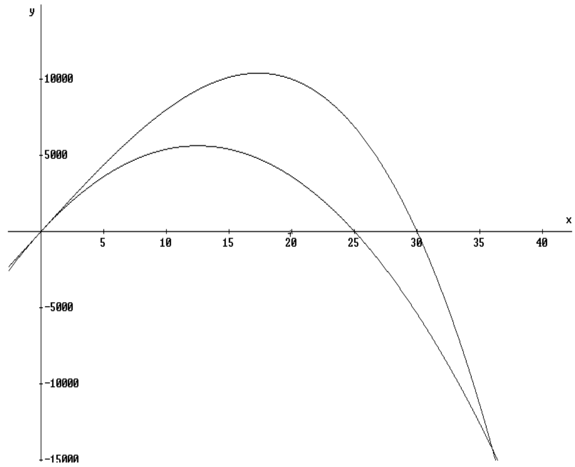
(ii) Prove that $\ddot{x} = 4(5 - x)$.

(iii) Find the smallest value of t for which the acceleration is zero.

Question 4

The graph shows the positions of two objects A and B at time t are given by

$$x_A = 900t - t^3$$
$$x_B = 900t - 36t^2$$



(i) Find the time interval in which the position of each object is to the right of the origin.

(ii) Find an expression (in terms of t) for the distance (D) between the two objects while they are both to the right of the origin

(iii) Find the greatest distance between the two objects while they are to the right of the origin.

Quadratics

Question 1

Find the roots of the equation $2y^2 - 6y + 3 = 0$. Give your answer in simplest form.

Question 2

(i) Sketch the parabola $y = 2x^2 + 5x - 12$ clearly showing all intercepts with the coordinate axes.

(ii) Hence or otherwise solve $2p^2 + 5p - 12 > 0$.

Question 3

The roots of the equation $3x^2 - 5x + 1 = 0$ are $x = \alpha$ and $x = \beta$. Find the value of

(i) $\alpha + \beta$.

(ii0 $\alpha\beta$.

(iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 2$.

Question 4

(i) If the line $y = mx + b$ is tangent to the hyperbola $y = \frac{a}{x}$ prove that $b^2 + 4am = 0$.

(ii) Hence find the values of b for which the line $y = b - 3x$ always intersects the hyperbola $y = \frac{12}{x}$.

Question 5

(i) Prove that $(m-n)^2 = (m+n)^2 - 4mn$.

(ii) Given that $x = \alpha$ and $x = \beta$ are roots of the quadratic equation $px^2 + qx + 1 = 0$ and $\alpha > \beta$, find an expression for $\alpha - \beta$. Express your answer in simplest form.

Question 6

Find the value(s) of *k* for which *********** is positive definite

Question 7

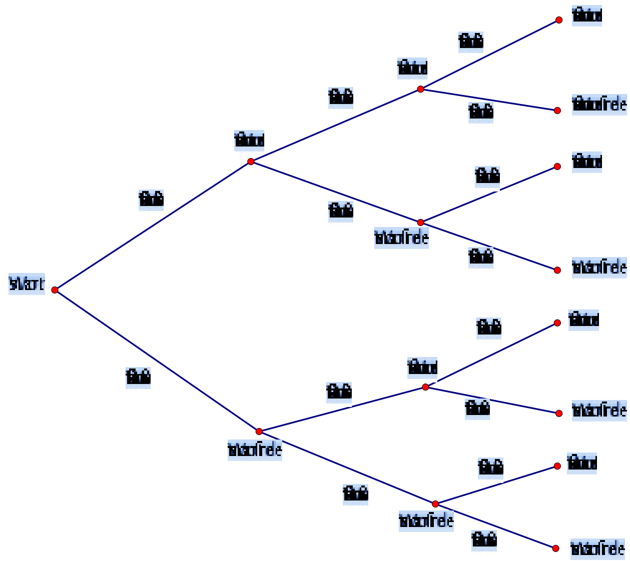
Show that ************** always has rational roots if ********* are rational.

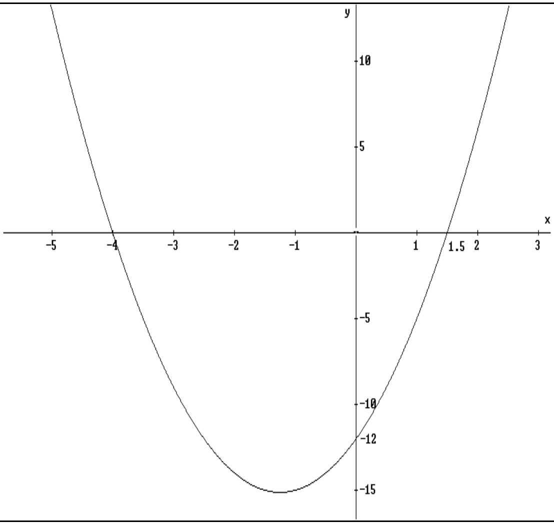
Question 8

Find the minimum value of *y* if $y = 3x^2 - 8x + 6$.

Question 1

(a)	(i)	when $t = 0$, $2916 = P_0 e^0$ $\therefore\ P_0 = 2916$ when $t = 10$, $4860 = 2916e^{10k}$ $e^{10k} = \frac{5}{3}$ $10k = \ln\left(\frac{5}{3}\right)$ $k = \frac{1}{10}\ln\left(\frac{5}{3}\right)$	3
	(ii)	when $t = 40$, $P = 2916e^{4\ln\left(\frac{5}{3}\right)}$ $\therefore$ population $= 22\ 500$	2
	(iii)	$\frac{dP}{dt} = kP_0 e^{kt}$ $= kP$ since $P = P_0 e^{kt}$	1
	(iv)	(α) when $P = 600$, $\frac{dP}{dt} = \frac{1}{10}\ln\left(\frac{5}{3}\right)\times 600$ $= 310$ (to 2 sig. fig.)	1
		(β) when $t = 30$, $P = kP_0 e^{kt}$ $= \frac{1}{10}\ln\left(\frac{5}{3}\right)\times 2916 \times e^{3\ln\left(\frac{5}{3}\right)}$ $= 690$ (to 2 sig. fig.)	1
(b)	(i)	when $x_A = 0$, $t(30-t)(30+t) = 0$ $t_A = 0 \text{ or } \pm 30$ when $x_B = 0$, $36t(25-t) = 0$ $t_B = 0 \text{ or } 25$ Time intervals are $0 < t_A < 30$ and $0 < t_B < 25$.	2
	(ii)	$D = x_A - x_B$ $= (900t - t^3) - (900t - 36t^2)$ $D = 36t^2 - t^3$	1

	(iii)	$\frac{dD}{dt}=72t-3t^2$ for stat. pt. $\frac{dD}{dt}=0$ $72t-3t^2=0$ $3t(24-t)=0$ $t=0$ or 24 $\frac{d^2D}{dt^2}=72-6t$ when $t=0,\ \frac{d^2D}{dt^2}=72>0$ $\therefore$ local min. tp. when $t=24,\ \frac{d^2D}{dt^2}=-72<0$ $\therefore$ local max. tp. when $t=24,\ D=36(24)^2-(24)^3$ $=6912$ maximum distance is 6912 m.	4
Question 2			
(a)	(i)		1
	(ii)	$P(\text{no free Mars bar})=(0.8)^3$ or $\left(\frac{4}{5}\right)^3$ $=0.512$ or $\frac{64}{125}$	1
	(iii)	$P(\text{at laest one free Mars bar})=1-P(\text{no free Mars bars})$ $=1-(0.8)^3$ $=0.488$ or $\frac{61}{125}$	1
	(iv)	$P(\text{one free Mars bar})=3\times(0.2)\times(0.8)^2$ or $3\times\left(\frac{1}{5}\right)\times\left(\frac{4}{5}\right)^2$ $=0.384$ or $\frac{48}{125}$	1

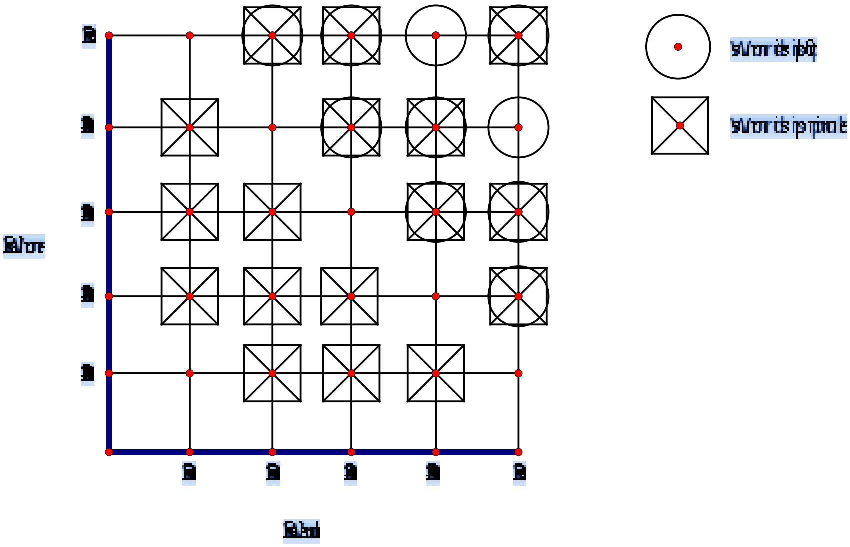
(b)	(i)	if $x = 0,\ y = -12$ if $y = 0,\ 2x^2 + 5x - 12 = 0$ $(2x-3)(x+4) = 0$ $x = \frac{3}{2}\ or - 4$  	2
	(ii)	$\{p : p < -4\} \cup \{p : p > 1.5\}$ or $p < -4$ or $p > 1.5$	1
(c)	(i)	Since the expression for velocity is a quadratic with negative leading coefficient, the maximum speed is at the turning point $\therefore\ t = -\frac{10}{-2}$ $= 5$ $v = 24 + 10(5) - (5)^2$ $= 49$ $\therefore$ max speed = 49 *km/hr*	2
	(ii)	at rest when $v = 0$ $-t^2 + 10t + 24 = 0$ $t^2 - 10t - 24 = 0$ $(t-12)(t+2) = 0$ $t = 12 \quad (t \geq 0)$ $a = 10 - 2t$ when $t = 12, \quad a = 10 - 24$ $= -14$ $\therefore$ acceleration is $-14\ km/hr^2$	3

	(iii)	$x = 24t + 5t^2 - \frac{1}{3}t^3 + c$ when $t = 0, \quad x = 0$ $0 = 0 + 0 + 0 + c$ $\therefore c = 0$ $x = 24t + 5t^2 - \frac{1}{3}t^3$ when $t = 0, \quad x = 0$ when $t = 12, \quad x = 24(12) + 5(12)^2 - \frac{1}{3}(12)^3$ $= 432$ when $t = 15, \quad x = 24(15) + 5(15)^2 - \frac{1}{3}(15)^3$ $= 360$ distance travelled $= 432 + (432 - 360)\ km$ $= 504\ km$	3

Question 3

(a)	(i)	At 2 hours	1
	(ii)	At 80m to the right of the origin or at position B	1
	(iii)	$0 \le t < 4$	2
	(iv)	The object is moving to the left with increasing speed.	2
(b)	(i)	when $t = 0, \quad N = 12\ 000$ $12\ 000 = \frac{A}{5 + 3e^0}$ $A = 96\ 000$	1
	(ii)	when $t = 8$ $N = \frac{96000}{5 + 3e^{-0.1 \times 8}}$ ≈ 15122.9 $= 15100$ (to nearest 100)	1
	(iii)	when $N = 18000$ $18000 = \frac{96000}{5 + 3e^{-0.1t}}$ $5 + 3e^{-0.1t} = \frac{16}{3}$ $3e^{-0.1t} = \frac{1}{3}$ $e^{-0.1t} = \frac{1}{9}$ $-0.01t = \ln\left(\frac{1}{9}\right)$ $t = \frac{\ln\left(\frac{1}{9}\right)}{-0.1}$ $= 21.97$ time $= 22.0$ years (to 1 d.p.)	2

	(iv)	as $t \to \infty$, $e^{-kt} \to 0$ $\therefore\ N \to \frac{96000}{5}$ $N \to 19\ 200$ population tends to19200 penguins.	1
(c)	(i)	Curves meet when $mx + b = \dfrac{a}{x}$ $mx^2 + bx = a$ $mx^2 + bx - a = 0$ Now for the line to be a tangent, this equation has must have only one solution i.e. the $\Delta = 0$ $\therefore\ \Delta = b^2 - 4m(-a)$ $b^2 - 4m(-a) = 0$ $b^2 + 4ma = 0$	2
	(ii)	Let $m = -3$ and $a = 12$ $\therefore b^2 + 4(12)(-3) \geq 0$ $b^2 - 144 \geq 0$ $(b-12)(b+12) \geq 0$ $b \leq -12$ or $b \geq 12$	2
Question 4			
(a)	(i)	$\alpha + \beta = \frac{5}{3}$	1
	(ii)	$\alpha\beta = \frac{1}{3}$	1
	(iii)	$\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta}$ $= \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$ $= \dfrac{\left(\frac{5}{3}\right)^2 - 2\left(\frac{1}{3}\right)}{\left(\frac{1}{3}\right)}$ $= \frac{19}{3}$	2

(b)	(i)		1
	(ii)	$P(\text{sum} > 10) = \frac{10}{25}$ $= \frac{2}{5}$	1
	(iii)	$P(\text{sum is prime}) = \frac{17}{25}$	1
	(iv)	$P(\text{sum} > 10 \text{ given the sum is prime}) = \frac{8}{17}$	2
(c)	(i)	$\ddot{x} = -(5t+1)^{-2}$ $v = \dfrac{-(5t+1)^{-1}}{-5} + c_1$ $= \dfrac{0.2}{5t+1} + c_1$ when $t = 0, v = 1$ $\therefore\ 1 = 0.2 + c_1$ $c_1 = 0.8$ $v = \dfrac{0.2}{5t+1} + 0.8$	2
	(ii)	as $t \to \infty$ $v \to 0 + 0.8$ limiting speed is 0.8 m/s	1

	(iii)	$x = \frac{0.2\ln(5t+1)}{5} + 0.8t + c_2$ $= 0.04\ln(5t+1) + 0.8t + c_2$ when $t = 0,\ x = -3$ $-3 = 0.04\ln(1) + 0 + c_2$ $c_2 = -3$ $x = 0.04\ln(5t+1) + 0.8t - 3$ when $t = 10$ $x = 0.04\ln(51) + 8 - 3$ ≈ 5.157 position is 5m to the right of the origin.	3